Complex Analysis Lecture Notes (2024/2025)

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1 Preliminaries

1.1 Complex numbers

Algebraic operations on $\mathbb{C}$

$$(a+bi)+(c+di)=(a+c)+(b+d)i \qquad (a+bi)(c+di)=(ac-bd)-(ad-bc)i$$

$$(x+yi)^{-1}=\frac{1}{x+yi}\frac{x-yi}{x-yi}=\frac{x-yi}{x^2-y^2} \qquad |x+yi|=\sqrt{x^2+y^2} \qquad \overline{x+yi}=x-yi$$

Polar representation of complex numbers

$$z=x+yi=r(\cos\varphi+i\sin\varphi)=re^{i\varphi} \qquad z_1z_2=r_1r_2e^{i(\varphi_1+\varphi_2)}$$

Complex roots

$$z^n=re^{i\varphi} \implies z=r^{\frac{1}{n}}e^{\frac{i\varphi}{n}+\frac{i2\pi k}{n}} \qquad k\in\{0,1,\dots,n-1\}$$

Theorem *Fundamental theorem of algebra*

The polynomial equation $a^nz_n+a^{n-1}z_{n-1}+\ldots+a_0=0$ has exactly n solutions in $\mathbb{C}$.

$\mathbb{C}$ as a vector space

Multiplication by $x+yi$ can be represented by the matrix $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$

All $\mathbb{C}$-linear transformations are of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

1.2 Sequences and series

$\mathbb{C}$ is a metric space with the metric $d(z_1,z_2)=|z_1-z_2|$.

Definition *Convergent series*

The series $\sum_{k=0}^{\infty} z_k$ is **convergent** if its sequence of partial sums converges.

The series $\sum_{k=0}^{\infty} z_k$ is **absolutely convergent** if $\sum_{k=0}^{\infty} |z_k|$ converges.

Proposition

$z_n=x_n+iy_n$ converges to $z=x+iy \iff x_n$ converges to x and y_n converges to y

Proposition *Convergence tests*

Ratio test: $L=\lim_{k\to\infty}\frac{|z_{k+1}|}{|z_k|}$ **Root test**: $L=\lim_{k\to\infty}|z_k|^{\frac{1}{k}}$

Root test (stronger version, limit always exists): $L=\limsup_{k\to\infty}|z_k|^{\frac{1}{k}}$

If $L<1$ then $\sum_{k=0}^{\infty} z_k$ converges absolutely. If $L>1$ then $\sum_{k=0}^{\infty} z_k$ diverges.
If $L=1$ then the test is inconclusive.

Definition *Power series*

For the **power series** $f(z)=\sum_{k=0}^{\infty} c_k(z-a)^k$ we define the **radius of convergence** as $R=\frac{1}{\limsup|c_k|^{\frac{1}{k}}}$.
The power series converges for all z with $|z-a|<R$.

1.3 Continuity of complex functions

We usually require the domain U of a complex function to be open and connected.

Definition *Continuous function*

A complex function $f : U \to V$ is **continuous** at z_0 if:

$$\text{for all } \varepsilon > 0 \quad \text{there exists } \delta > 0 \quad \text{such that} \quad |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$$

Proposition

Polynomials are continuous on $\mathbb{C}$. Power series are continuous within their radius of convergence.

Proposition

$$f(x + yi) = u(x, y) + iv(x, y) \text{ is continuous} \iff u(x, y) \text{ and } v(x, y) \text{ are continuous}$$

2 Holomorphic functions

2.1 Complex derivatives

Definition *Complex differentiable function*

A complex function $f : U \to V$ is **complex differentiable** at z_0 if the limit $f'(a) = \lim_{z \to a} \frac{f(z) - f(z_0)}{z - z_0}$ exists.
f is **holomorphic** if it is complex differentiable at every point $z_0 \in U$.
f is **entire** if it is a holomorphic function $\mathbb{C} \to \mathbb{C}$.

Definition *Real differentiable function*

$f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is **real differentiable** if all of its partial derivatives exist and are continuous.

Theorem *Cauchy-Riemann equations*

If f is complex differentiable at a, then f satisfies the **Cauchy-Riemann equations** at a:

$$u_x(a) = v_y(a) \qquad v_x(a) = -u_y(a)$$

where $\begin{bmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(x) \end{bmatrix}$ is the Jacobian matrix at a when f is viewed as a function $U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$.

Theorem

$$f : U \to \mathbb{C} \text{ is holomorphic} \iff f \text{ is } \mathbb{R}\text{-differentiable and satisfies the Cauchy-Riemann equations } \forall a \in U$$

2.2 Uniform convergence

Definition *Open disk*

$$D_r(a) := \{z : |z - a| < r\}$$

Definition *Uniform convergence*

A sequence of functions $\{f_n\}$ **converges uniformly** to f on $U \subseteq \mathbb{C}$ if:

$$\forall \varepsilon > 0 \;\; \exists N \quad \text{such that } |f_n(z) - f(z)| < \varepsilon \quad \text{for all } n \leq N$$

Lemma

If all f_n are continuous on U and $f_n \to f$ uniformly, then f is continuous on U.

Lemma *Weierstrass M-test*

If $|f_n(z)| \leq M_n$ for all $z \in U$ and $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{k=0}^{\infty} f_k(z)$ converges uniformly in U.

Lemma

If $\sum_{k=0}^{\infty} a_k$ converges then $\lim_{k \to \infty} a_k = 0$

Theorem *Cauchy-Hadamard theorem*

$\sum_{k=0}^{\infty} c_k z^k$ converges absolutely for all z with $|z| < R$ and diverges for $|z| > R$, where $R = \dfrac{1}{\limsup |c_k|^{\frac{1}{k}}}$.
Moreover, the convergence is uniform on $D_s(0)$ for all $0 \leq s < R$.

Theorem

$f(z) = \sum_{k=0}^{\infty} c_k z^k$ is $\mathbb{C}$-differentiable at each z_0 with $|z - z_0| < R$, where $R = \dfrac{1}{\limsup |c_k|^{\frac{1}{k}}}$.
The derivative is given by $f'(z) = \sum_{k=1}^{\infty} k c_k z^{k-1}$.

2.3 Inverse functions

Theorem *Inverse function theorem*

If U and V are open subsets of $\mathbb{C}$, $f : U \to V$ is holomorphic and injective with continuous inverse $g : V \to U$ and $f'(z) \neq 0$ for all $z \in U$, then g is holomorphic with

$$g'(z_0) = \frac{1}{f'(g(z_0))} \quad \text{for all } z_0 \in V$$

Definition *Complex logarithm*

We can make the exponential function injective by restricting it to $\mathcal{U}_\alpha := \{z : \alpha < \operatorname{Im}(z) < \alpha + 2\pi\}$
The codomain of this injective exponential function is $\mathcal{V}_\alpha := \mathbb{C} \setminus \{\lambda e^{i\alpha} : \lambda \in \mathbb{R}, \lambda > 0\}$
An α-**branch** of the **complex logarithm** is the function $\ln_\alpha : \mathcal{V}_\alpha \to \mathcal{U}_\alpha$ defined by $g(z) = \ln|z| + i\arg(z)$
When we choose $\alpha = -\pi$, this gives us the **principal branch** of the logarithm.
A power of a complex number depends on the chosen branch: $z^w = e^{w \ln z}$

Trigonometric functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\arccos z = -i\ln(z + (z^2 - 1)^{\frac{1}{2}}) \qquad \arcsin z = -i\ln(iz + (1 - z^2)^{\frac{1}{2}})$$

3 Path integrals

Definition *Smooth curve*

Let C be an oriented curve in $\mathbb{C}$ and let $\gamma : [a, b] \to \mathbb{C}$ be a parametrization of C: $\gamma(t) = x(t) + iy(t)$
If $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ exist and are continuous, and $\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} \neq 0 \;\; \forall t$, then C is a **smooth curve**.

Definition *Path integral*

Let C be a smooth curve and $f(z)$ a continuous function on C. Then the **path integral** of $f(z)$ on C is:

$$\int_C f(z)\,\mathrm{d}z = \int_{t=a}^{b} f(\gamma(t))\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,\mathrm{d}t = \int_{t=a}^{b} f(x(t), y(t)) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t} + i\frac{\mathrm{d}y}{\mathrm{d}t}\right)\,\mathrm{d}t$$

Proposition

1. Path integrals are independent of the parametrization.
2. Path integrals depend on the orientation of the curve: $\int_{-C} f(z)\,\mathrm{d}z = -\int_C f(z)\,\mathrm{d}z$

Proposition *Algebraic properties of path integrals*

$$\int_C f(z)+g(z)\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z + \int_C g(z)\,\mathrm{d}z \qquad \int_C cf(z) = c\int_C f(z)\,\mathrm{d}z$$

Definition *Length of a curve*

The **length** of a curve C is $\int_{t=a}^{b} |\mathrm{d}z|\,\mathrm{d}t = \int_{t=a}^{b} \left(\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right)^{\frac{1}{2}} \mathrm{d}t$

Lemma

Let $G : [a,b] \to \mathbb{C}$. Then $\left|\int_{t=0}^{b} G(t)\,\mathrm{d}t\right| \leq \int_{t=0}^{b} |G(t)|\,\mathrm{d}t$

Proposition *ML-inequality*

If $|f(z)| \leq M$ for all $z \in C$, then $\left|\int_C f(z)\,\mathrm{d}z\right| \leq ML$, where L is the length of C.

Proposition

If $f_n(z)$ converges uniformly to $f(z)$ on C, then $\lim_{n\to\infty} \int_C f_n(z)\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z$

3.1 Primitives

Proposition

Suppose f is the derivative of some holomorphic function F on some open set containing the (piecewise) smooth curve C connecting point A to point B. Then

$$\int_C f(z)\,\mathrm{d}t = F(B) - F(A)$$

Definition *Primitive*

We say $f(z)$ admits a **primitive** in U if there is a holomorphic function $F(z)$ such that $F'(z) = f(z)$ for all $z \in U$.

Corollary

If f admits a primitive in U, then $\int_C f(z)\,\mathrm{d}z = 0$ for every closed curve C in U.

Lemma

Every linear polynomial $f(z) = \alpha + \beta z$ admits a primitive.

3.1.1 Primitives on disk-like domains

Lemma *Rectangle lemma (Goursat)*

Let R be a rectangle and let $f(z)$ be a holomorphic function on some open set U containing R. Then $\int_{\partial R} f(z) = 0$.

Definition *Disk-like domain*

An open set $D \subseteq \mathbb{C}$ is a **disk-like domain** if for all pairs of points $z, w \in D$, the rectangle with vertices z, $\mathrm{Re}(w) + i\,\mathrm{Im}(z)$, w, $\mathrm{Re}(z) + i\,\mathrm{Im}(w)$ is contained in D

Notation

$$\int_a^b f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z$$

where γ is the line segment from a to $\mathrm{Re}(a) + i\,\mathrm{Im}(b)$ followed by the line segment from $\mathrm{Re}(a) + i\,\mathrm{Im}(b)$ to b.

Proposition

If f is holomorphic on a disk-like domain D and $a \in D$, then f admits a primitive:

Define $F(z) = \displaystyle\int_a^z f(w)\,\mathrm{d}w$. Then F is holomorphic on D with $F'(z) = f(z)$.

Corollary *Closed curve theorem*

If f is holomorphic on a disk-like domain D and C is a closed curve in D, then $\displaystyle\int_C f(z)\,\mathrm{d}z = 0$

Proposition

If D is a disk-like domain, then for any holomorphic function f on D $\displaystyle\int_a^z f(w)\,\mathrm{d}w = \int_\Gamma f(w)\,\mathrm{d}w$

where Γ is any piecewise smooth curve from a to z.

3.1.2 Primitives on simply connected sets

Definition *Homotopic paths*

Let $\gamma_0 : [0,1] \to D$ and $\gamma_1 : [0,1] \to D$ be (piecewise) smooth paths with the same end points:

$$\gamma_0(0) = a \qquad \gamma_1(0) = a \qquad \gamma_0(1) = b \qquad \gamma_1(1) = b$$

γ_0 and γ_1 are **homotopic** is there exists a continuous map $H : [0,1] \times [0,1] \to D$ such that

$$H(0,t) = \gamma_0(t) \qquad H(1,t) = \gamma_1(t) \qquad H(s,0) = a \qquad H(s,1) = b \qquad \text{for all } s,t \in [0,1]$$

Theorem

If γ_0 and γ_1 are homotopic paths, and f is holomorphic in U, then

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

Definition *Simply connected set*

$D \subseteq \mathbb{C}$ is **simply connected** if it is path-connected and any two paths with the same endpoints in D are homotopic.

Theorem

Suppose $D \subseteq \mathbb{C}$ is open and simply connected and f is a holomorphic function on D.

Then for any path γ from a to z, $\displaystyle\int_\gamma f(w)\,\mathrm{d}w$ is independent of the path and gives a well-defined holomorphic function F on D with $F'(z) = f(z)$.

Corollary

If f is holomorphic on an open simply connected set $U \subseteq \mathbb{C}$, then $\displaystyle\int_C f(z)\,\mathrm{d}z = 0$ for any closed curve C in U.

Proposition

Let $D \subseteq \mathbb{C}$ be open and simply connected, $z_0 \in D$ and $g(z) := \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0 \\ f'(z_0) & \text{if } z = z_0 \end{cases}$

Goursat's rectangle lemma and the closed curve theorem hold for $g(z)$.

3.2 Cauchy integral formula

Theorem *Cauchy integral formula*

Let f be a holomorphic function on an open simply connected set $U \subseteq \mathbb{C}$.
Let C be a simple closed curve oriented counter-clockwise in U and let a be a point in the interior of C. Then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - a} \, \mathrm{d}z \qquad \text{or alternatively:} \quad \int_C \frac{f(z)}{z - a} \, \mathrm{d}z = 2\pi i \cdot f(a)$$

4 Analytic functions

Definition *Analytic function*

A function $f(z)$ is an **analytic function** on $D_R(a)$ if it is defined by a power series:

$$f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k \qquad R = \frac{1}{\limsup |c_k|^{\frac{1}{k}}}$$

Theorem

Suppose f is holomorphic on a disk $D_R(a)$ for some $a \in \mathbb{C}$ and $R > 0$.
Then there exist unique constants $c_1, c_2, \ldots$ such that

$$f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k \qquad \text{for all } z \in D_R(a)$$

Theorem

$$f \text{ is analytic on } D_R(a) \iff f \text{ is holomorphic on } D_R(a)$$

Corollary

The derivative of a holomorphic function is holomorphic.

4.1 Uniqueness of holomorphic functions

Theorem *Liouville's theorem*

If $f(z)$ is entire (holomorphic on $\mathbb{C}$) and bounded, then $f(z)$ is constant.

Proposition

Let $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$ be a power series on an open disk $D_R(0)$ and let $\{z_n\}$ be a sequence of nonzero complex numbers in $D_R(0)$ with $\lim\limits_{n \to \infty} z_n = 0$. Assume $f(z_n) = 0$ for all n. Then $f(z) = 0$ for all $z \in D_R(0)$.

Proposition

Suppose U is an open and connected subset of $\mathbb{C}$ and $f(z)$ is a holomorphic function on U.
Assume there is a sequence of pairwise distinct complex numbers $\{z_n\}$ in U such that $f(z_n) = 0$ for all n and $\lim\limits_{n \to \infty} z_n \in U$. Then $f(z) = 0$ for all $z \in U$.

Theorem *Uniqueness theorem*

Let $U \subseteq \mathbb{C}$ be open and connected and $A \subseteq U$ an infinite set with at least 1 accumulation point in U.
If f and g are holomorphic on U such that $f(z) = g(z)$ for all $z \in A$, then $f(z) = g(z)$ for all $z \in U$.

4.2 Maximum and minimum modulus

Theorem *Mean value theorem*

Suppose $f(z)$ is holomorphic on an open set containing the closed disk $D_\rho(a)$. Then

$$f(a) = \frac{1}{2\pi} \int_{\rho=0}^{2\pi} f(a + \rho e^{i\varphi}) \, \mathrm{d}\varphi$$

Theorem *Maximum modulus theorem*

Suppose $D \subseteq \mathbb{C}$ is open and connected and f is a non-constant holomorphic function on D.
Then $|f|$ does not admit any local maximum on D.

Corollary

If $K \subseteq \mathbb{C}$ is a compact set and f is non-constant, continuous on K and holomorphic on the interior of K, then $|f|$ attains its maximum on ∂K.

Theorem *Minimum modulus theorem*

Suppose D is open and connected, and $|f|$ is holomorphic and has a local minimum at $z_0 \in D$.
Then f is constant or $|f(z_0)| = 0$.

Theorem *Open mapping theorem*

The image of an open set under a non-constant holomorphic function is open.

4.3 Morera's theorem

Theorem *Morera's theorem*

Let f be a continuous function on an open set $U \subseteq \mathbb{C}$.
If $\int_{\partial R} f(z) = 0$ for all rectangles $R \subseteq U$ whose sides are parallel to the coordinate axes, then f is holomorphic in U.
(Most textbooks show a weaker statement where R is any arbitrary closed curve)

Theorem

Suppose $\{f_n\}$ is a sequence of holomorphic functions on an open set U.
If f_n uniformly converges to f on every compact subset of U, then f is also a holomorphic function.

Lemma *Schwarz's lemma*

Let D be the open disk of radius 1 around the origin. Let $f : D \to D$ be holomorphic with $f(0) = 0$.
Then f satisfies the following:

1. $|f(z)| \leq |z|$ for all $z \in D$
2. $|f'(0)| \leq 1$
3. Equality holds for 1) and 2) if and only if $f(z) = e^{i\theta_0} z$ for some fixed θ_0

5 Singularities

5.1 Holomorphic functions on line segments

Theorem

Suppose f is continuous on an open set $D \subseteq \mathbb{C}$ and holomorphic on D except possibly on a line segment $L \subseteq D$. Then f is holomorphic throughout D.

Theorem *Schwartz's reflection principle*

Let U be an open connected subset of the upper half of the complex plane, such that $\partial U \cap \mathbb{R} = L$ is a line segment.
Let $U^* := z : \overline{z} \in U$ be the reflection of U across the real axis.
Suppose that $f(z)$ is holomorphic on U and continuous on $U \cup L$, such that $f(z) \in \mathbb{R}$ for all $z \in L$.
Then we can define a holomorphic extension g of f for $U \cup L \cup U^*$:

$$g(z) = \begin{cases} f(z) & \text{if } z \in U \cup L \\ \overline{f(\overline{z})} & \text{if } z \in U^* \end{cases}$$

5.2 Singularities

Definition *Deleted neighborhood*

$$D_r^\circ(z_0) := D_r(z_0) \setminus \{z_0\}$$

Definition *Isolated singularity*

We say f has an **isolated singularity** if there is a deleted neighborhood $D_r^\circ(z_0)$ such that f is holomorphic in $D_r^\circ(z_0)$ but f is not holomorphic on $D_r(z_0)$. (Note that f is necessarily discontinuous at z_0.)

Definition *Classification of isolated singularities*

Suppose f has an isolated singularity at z_0.

1. If there exists a holomorphic function g in $D_r(z_0)$ such that $g(z) = f(z)$ for all $z \in D_r^\circ(z_0)$, then f has a **removable singularity** at z_0.
2. If there exist holomorphic functions $A(z)$ and $B(z)$ in $D_r(z_0)$ such that

   $$A(z_0) \neq 0 \qquad B(z_0) = 0 \qquad f(z) = \frac{A(z)}{B(z)} \qquad \text{for all } z \in D_r^\circ(z_0)$$

   then f has a **pole** at z_0. The **order** of a pole is the multiplicity of z_0 as a root of $B(z)$.
3. If f has neither a removable singularity nor a pole at z_0, then f has an **essential singularity** at z_0

Theorem *Riemann's principle for removable singularities*

If f has an isolated singularity at z_0 and $\lim_{z \to z_0} (z - z_0) f(z) = 0$ then the singularity is removable.

Corollary

If f has an isolated singularity at z_0 and f is bounded in a deleted neighborhood of z_0, then the singularity is removable.

Theorem

Suppose f has an isolated singularity at z_0 and $\lim_{z \to z_0} (z - z_0)^k f(z) \neq 0$ but $\lim_{z \to z_0} (z - z_0)^{k+1} f(z) = 0$.
Then f has a pole of order k at z_0.

Corollary

If f has a pole of order k at z_0, then $\lim_{z \to z_0} |f(z)| = \infty$

Theorem *Casorati-Weierstrass theorem*

If f has an essential singularity at z_0, then the range $R = \{f(z) : z \in D_r^\circ(z_0)\}$ is dense in $\mathbb{C}$.

Theorem *Picard's theorem*

If f has an essential singularity at z_0, then the range $R = \{f(z) : z \in D_r^\circ(z_0)\}$ is either $\mathbb{C}$ or $\mathbb{C} \setminus p$ for some point p.

5.3 Laurent series

Definition *Two-sided series*

The **two-sided series** $\sum_{k=-\infty}^{\infty} u_k$ converges to L if the following are true:

1. $\sum_{k=0}^{\infty} u_k$ converges to L_1
2. $\sum_{k=1}^{\infty} u_{-k}$ converges to L_2
3. $L_1 + L_2 = L$

Definition *Laurent series*

A two-sided power series is called a **Laurent series**.

$$f_1(z) = \sum_{k=1}^{\infty} c_{-k} \left(\frac{1}{z}\right)^k \qquad f_2(z) = \sum_{k=0}^{\infty} c_k z^k \qquad f(z) = f_1(z) + f_2(z) = \sum_{k=-\infty}^{\infty} c_k z^k$$

We call $f_1(z)$ the **principal part** and $f_2(z)$ the **analytic part** of the Laurent series.

Proposition

Suppose that $f(z)$ is defined by a Laurent series with the following radii of convergence:

$$R_1 = \limsup |c_{-k}|^{\frac{1}{k}} \text{ (principal part)} \qquad R_2 = \frac{1}{\limsup |c_k|^{\frac{1}{k}}} \text{ (analytic part)}$$

Suppose that $R_1 < R_2$. Then $f(z)$ is a holomorphic function on the annulus $R_1 < |z| < R_2$.

Theorem

If f is holomorphic on the annulus $A := \{z : R_1 < |z - z_0| < R_2\}$, then f has a unique Laurent series expansion:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k \qquad c_k = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}} \, \mathrm{d}w$$

where C is any circle inside A.

Proposition *Laurent series around singularities*

Let f be a holomorphic function and c_k the coefficients of its Laurent series.

1. If f has a removable singularity at z_0, then the principal part of its Laurent series at z_0 is 0.
2. If f has a pole of order k at z_0, then $c_{-k} \neq 0$ and $c_{-N} = 0$ for all $N > k$.
3. If f has an essential singularity at z_0, then the principal part of its Laurent series at z_0 contains infinitely many nonzero terms.

Theorem *Partial fraction decomposition theorem*

Suppose

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z-z_1)^{k_1}\dots(z-z_n)^{k_n}}$$

where P, Q are polynomials, $z_1, \dots, z_n$ are pairwise distinct and $\deg(P) < \deg(Q)$.
Then $R(z)$ can be expanded as a sum of polynomials in $\frac{1}{z-z_j}$, where z_j are roots of $Q(z)$.

$$R(z) = \left(\frac{a_1}{z-z_1}\right) + \left(\frac{a_2}{z-z_1}\right)^2 + \left(\frac{a_3}{z-z_1}\right)^3 + \dots + \left(\frac{b_1}{z-z_2}\right) + \left(\frac{b_2}{z-z_2}\right)^2 + \left(\frac{b_3}{z-z_2}\right)^3 + \dots$$

5.4 Residues

Definition *Residue*

If f has an isolated singularity at z_0, we call the coefficient c_{-1} of the Laurent series the **residue** of f at z_0.
We denote the residue by $\operatorname{Res}(f; z_0)$.

Proposition

If $f(z) = \frac{A(z)}{B(z)}$ (with $A(z), B(z)$ holomorphic) has a **simple pole** (a pole of order 1) at z_0, then

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z) = \frac{A(z_0)}{B'(z_0)}$$

Corollary

If $f(z)$ has a pole of order k at z_0, then

$$\operatorname{Res}(f, z_0) = \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\left((z-z_0)^k f(z)\right)$$

5.5 Winding numbers

Definition

Let γ be a closed curve in $\mathbb{C}$ and $a \notin \gamma$. Then

$$W(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$

is called the **winding number** of γ around a.

The winding number $W(\gamma, a)$ is the number of times that γ winds around a with counterclockwise orientation.

Proposition

If γ is a circle oriented counterclockwise, and a is at the center of γ, then $W(\gamma, a) = 1$.

Proposition

If γ is a simple closed curve in $\mathbb{C}$ oriented counter-clockwise, then $W(\gamma, a) = \begin{cases} 1 \text{ if } a \text{ is inside } \gamma \\ 0 \text{ if } a \text{ is outside } \gamma \end{cases}$

Theorem

For any closed curve γ and any $a \notin \gamma$, $W(\gamma, a)$ is an integer.

Corollary

For any closed curve γ, W is constant on each connected component of $\mathbb{C} \setminus \gamma$

5.6 Cauchy's residue theorem

Theorem *Cauchy's residue theorem*

Suppose $U \subseteq \mathbb{C}$ is open and simply connected, and f is holomorphic in U except possibly at isolated points $z_1, \dots, z_n$. If γ is a closed curve in $U \setminus \{z_1, \dots, z_n\}$, then

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} W(\gamma, z_k)\,\mathrm{Res}(f, z_k)$$

Corollary

With u, f, γ as in the previous theorem, if γ is a simple closed curve oriented counterclockwise, then:

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$$

where the sum is over all singular points $z_1, \dots, z_n$ that are inside the curve.

Theorem *Generalized Cauchy integral formula*

Let f be holomorphic on an open and simply connected subset $U \subseteq \mathbb{C}$. Let γ be a simple closed curve in U. Then for all points z inside γ:

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{k+1}}\,\mathrm{d}w$$

Theorem

Let $U \subseteq \mathbb{C}$ be open and connected and let γ be a simple closed curve in U. Let $f(z)$ be a **meromorphic** function (holomorphic everywhere except at poles) on U such that none of the poles or zeroes of f are on γ.

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \#\{\text{zeroes of } f\} - \#\{\text{poles of } f\} \quad \text{(both counted with multiplicity)}$$

Theorem *Argument principle*

Let $U \subseteq \mathbb{C}$ be open and connected and let γ be a simple closed curve in U.
Let $f(z)$ be a holomorphic function in U that is non-zero on γ.

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = W(f(\gamma), 0) = \#\{\text{zeros of } f \text{ inside } \gamma \text{ counted with multiplicity}\}$$

Theorem *Rouché's theorem*

Let $U \subseteq \mathbb{C}$ be open and simply connected and $\gamma \subseteq U$ a simple and closed curve.
Let f and g be holomorphic functions in U. Suppose f and g are nonzero on γ and $|f(z)| > |g(z)|$ on γ.

$$\#\{\text{zeros of } f+g \text{ inside of } \gamma \text{ counted with multiplicity}\} = \#\{\text{zeros of } f \text{ inside of } \gamma \text{ counted with multiplicity}\}$$

5.6.1 Applications of the residue theorem

Evaluating real rational integrals using residues

Let $P(x), Q(x)$ be polynomials with $\deg(Q) \geq \deg(P) + 2$, such that Q has no real roots.
Then the (real) integral $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x$ is equal to $\int_C \frac{P(z)}{Q(z)}\,\mathrm{d}z$, where C is a semicircle inside the top half of the complex plane, centered at the origin, whose radius approaches infinity. This is true because the integral over the arc of the semicircle converges to 0 as R goes to infinity. The path integral can be computed using the residue theorem.

The same method can be used for certain other functions, as long as the real integral is absolutely convergent and it can be proven that the path integral over the arc of the semicircle converges to 0 as the radius approaches infinity.

Proposition *Evaluating two-sided sums using residues*

Let f be a function with isolated singularities at $z_1, \ldots, z_m$ that is bounded by $\dfrac{A}{|z|^2}$ for large $|z|$.
Let $\phi = \pi\cot(\pi z)$ and $\psi = \pi\csc(\pi z)$. Then:

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_{k=1}^{m} \mathrm{Res}(f(z)\phi(z), z_k) \qquad \sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum_{k=1}^{m} \mathrm{Res}(f(z)\psi(z), z_k)$$

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